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2001 J. Phys. A: Math. Gen. 34 663

(http://iopscience.iop.org/0305-4470/34/3/325)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 663–672 www.iop.org/Journals/ja PII: S0305-4470(01)14365-9

Algebraic invariant curves of plane polynomial differential systems

Alexei Tsygvintsev

Section de Mathématiques, Université de Genève, 2-4, rue du Lièvre, CH-1211, Case postale 240, Switzerland

E-mail: Alexei.Tsygvintsev@math.unige.ch

Received 25 May 2000, in final form 5 December 2000

Abstract

We consider a plane polynomial vector field $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$ of degree $m > 1$. With each algebraic invariant curve of such a field we associate a compact Riemann surface with the meromorphic differential $\omega = \mathrm{d}x/P = \mathrm{d}y/Q$. The asymptotic estimate of the degree of an arbitrary algebraic invariant curve is found. In the smooth case this estimate has already been found by Cerveau and Lins Neto in a different way.

AMS classification scheme number: 34C07

Introduction

The study of plane polynomial vector fields goes back at least to Poincaré [12]. Recall that the second half of Hilbert's 16th problem [8] asks for an upper bound on the number of limit cycles of real plane polynomial vector fields. Notice that the class of invariant curves of the given planar system involves the class of its limit cycles. Of course, every limit cycle is also an invariant curve.

This paper is devoted to one aspect of this problem: to study algebraic invariant curves, i.e. defined by an algebraic equation $f(x, y) = 0$, where $f \in \mathbb{C}[x, y]$ is an arbitrary polynomial. The real part of the above curve, which turns out to be a limit cycle, is called the *algebraic* limit cycle. Until now only a few cases of algebraic limit cycles have been known, especially for quadratic plane systems [3]. It has been shown by Darboux [4] that if a given planar polynomial system of degree m has more than $2 + [m(m + 1)]/2$ algebraic invariant curves, then it admits a rational first integral.

In this paper we apply a new method connecting the problem of existence of algebraic invariant curves of plane polynomial vector fields of the form $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$ with the contemporary theory of Riemann surfaces. With each algebraic invariant curve of such a field we associate a compact Riemann surface C and a meromorphic differential $\omega = \mathrm{d}x/P = \mathrm{d}y/Q$.

Using this approach, in section 5 we find the asymptotic estimate of the degree of an arbitrary algebraic invariant curve (theorem 6). In the particular case we obtain the estimate for a degree of a nodal algebraic invariant curve (corollary 3). It is shown too that an arbitrary smooth algebraic invariant curve has a degree less than $m + 2$ (theorem 2) and that for an arbitrary algebraic invariant curve its genus is a linear function of the degree (theorem 5). These results have already been obtained (in a completely different way) in papers [1, 2].

1. The Darboux divisor and points at infinity

Consider the system of differential equations

$$\dot{x} = P(x, y) \qquad \dot{y} = Q(x, y) \qquad (x, y) \in \mathbb{C}^2 \tag{1}$$

where P, Q are polynomials of degree $m > 1$. We suppose that P and Q have not a common nonconstant polynomial factor and $P = \sum_{i=1}^{m} P_i$, $Q = \sum_{i=1}^{m} Q_i$, where P_i, Q_i are homogeneous polynomials of degrees $i = 0, \ldots, m$.

Let $C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$ be an invariant curve of (1). Without loss of generality we may suppose that $f \in \mathbb{C}[x, y]$ is irreducible. Then $\dot{f} = (P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y})_{f=0} \equiv 0$. As the ideal $\langle f \rangle$ is radical, then $\dot{f} \in \langle f \rangle$ and hence $\dot{f} = kf$, for some $k \in \mathbb{C}[x, y]$.

Definition 1. *The polynomial $f(x, y) \in \mathbb{C}[x, y]$ is called an algebraic partial integral of the system (1) if there exists a polynomial $k \in \mathbb{C}[x, y]$ such that*

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf. \tag{2}$$

The polynomial k is called a cofactor and has the form $k = \sum_{i=1}^{m-1} k_i$, where k_i are homogeneous polynomials of degrees $i = 0, \ldots, m - 1$. If $k \equiv 0$ then $f(x, y) = \text{const}$ is a first integral of the system (1).

Remark 1. It is easy to see that if $f(x, y)$ is reducible, i.e. $f = f_1^{m_1} \cdots f_l^{m_l}$, where $f_k \in \mathbb{C}[x, y]$, $k = 1, \ldots, l$, then polynomials f_k are again partial integrals of the system (1).

The polynomial f is a sum of its homogeneous parts $f = \sum_{i=a}^{n} f_i$, where f_i are homogeneous polynomials of degrees $i = 0, \ldots, n$ and $n = \deg(f)$.

Consider the homogeneous polynomial $R_{m+1}(x, y)$ of degree $m + 1$ defined by

$$R_{m+1}(x, y) = xQ_m(x, y) - yP_m(x, y) \tag{3}$$

where P_m and Q_m are higher homogeneous parts of the polynomials P and Q respectively. Let us suppose that R_{m+1} does not vanish identically, then it has $m+1$ zeros $D_i = [x_i : y_i] \in \mathbb{CP}^1$, $i = 1, \ldots, m + 1$. By the suitable rotation of variables x, y we can obtain $x_i y_i \neq 0$, $i = 1, \ldots, m + 1$. Hence, without loss of generality, $D_i = (1, z_i)$, $z_i \in \mathbb{C}^2$, $z_i \neq 0$, $i = 1, \ldots, m + 1$.

Definition 2. *The formal sum of points $D = \sum_{i=1}^{m+1} D_i$ is called the Darboux divisor of the differential system (1).*

Notice that the important role of the points D_i for polynomial vector fields first was observed by Darboux in 1878.

Let

$$V(x, y) = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

be the polynomial vector field on $\mathbb{C}^2$ corresponding to the system (1). Through the nonlinear change of variables

$$u = \frac{1}{x} \qquad v = \frac{y}{x} \qquad x \neq 0 \qquad (u, v) \in \mathbb{C}^2$$

and multiplying the induced vector field by u^{m-1} we obtain [5, 6]

$$\tilde{V}(u,v) = A(u,v)\frac{\partial}{\partial u} + B(u,v)\frac{\partial}{\partial v}$$
$$A(u,v) = -u^{m+1}P\left(\frac{1}{u},\frac{v}{u}\right)$$
$$B(u,v) = u^m\left[Q\left(\frac{1}{u},\frac{v}{u}\right) - vP\left(\frac{1}{u},\frac{v}{u}\right)\right]$$

where $\tilde{V}(u,v)$ represents the vector field of (1) near the line at infinity $L_\infty = \{u = 0\}$. The point $(0, v_0)$ where $\tilde{V}(0, v_0) = (0,0)$ is the singular point of $\tilde{V}(u,v)$. It is easy to see that $R_{m+1}(1, v_0) = 0$ and we obtain the following proposition.

Proposition 1. *The points $D_i = (1, z_i) \in D$, $i = 1, \ldots, m+1$, are the singular points at infinity of the system (1).*

The equation (2) turns into

$$A(u,v)\frac{\partial F}{\partial u} + B(u,v)\frac{\partial F}{\partial v} = K(u,v)F$$

where $F(u,v) = u^n f(\frac{1}{u}, \frac{v}{u}) = f_n(1,v) + uf_{n-1}(1,v) + \cdots = 0$ represents the curve C near L_∞ and

$$K(u,v) = u^{m-1}k\left(\frac{1}{u},\frac{v}{u}\right) - u^m nP\left(\frac{1}{u},\frac{v}{u}\right).$$

Let us show now that the Darboux divisor D contains all possible points at infinity of any algebraic invariant curve of the system (1).

Denote by $L_\infty = \{[x_i : y_i : 0] : (x,y) \subset \mathbb{CP}^1\} \subset \mathbb{CP}^2$ the line at infinity. Let I_f be a set of points at infinity of the algebraic curve C which correponds to the equation $f(x,y) = 0$, where $f(x,y)$ is an algebraic partial integral of the system (1).

Theorem 1. *$I_f \subset D$.*

Proof. By considering the right- and left-hand homogeneous parts of (2) we find

$$P_m\frac{\partial f_n}{\partial x} + Q_m\frac{\partial f_n}{\partial y} = k_{m-1}f_n \tag{4}$$

where f_n is the highest-order term of the polynomial $f = \sum_{i=a}^n f_i$ and k_{m-1} is the highest-order term of the cofactor $k = \sum_{i=1}^{m-1} k_i$.

To show $I_f \subset D$ we need to prove that if $f_n(x_0, y_0) = 0$ then $(x_0, y_0) \in D$ or

$$R_{m+1}(x_0, y_0) = 0 \tag{5}$$

where the polynomial R_{m+1} is defined by (3).

Consider the linear change of variables $(x,y) \to (u,v)$: $x = x_0 + u$, $y = y_0 + v$. The polynomial $f_n(x,y)$ turns into the polynomial $F(u,v) = f_n(x_0 + u, y_0 + v)$, which has the following Taylor expansion:

$$F(u,v) = \sum_{i=r}^{n} F_i(u,v) \tag{6}$$

where F_i are homogeneous polynomials of degrees $i = r, \ldots, n$, $r \geqslant 1$ and

$$F_i = \frac{1}{(n-i)!}\left(x_0\frac{\partial}{\partial u} + y_0\frac{\partial}{\partial v}\right)^{n-i} f_n(u,v).$$

Thus, for the lower-order term F_r of the sum (6) we have $F_r \not\equiv \text{const}$ and the following identity is fulfilled:

$$x_0 \frac{\partial F_r}{\partial u} + y_0 \frac{\partial F_r}{\partial v} = 0. \tag{7}$$

The equation (4) takes the form

$$\left((c_1 + N_1(u,v)) \frac{\partial}{\partial u} + (c_2 + N_2(u,v)) \frac{\partial}{\partial v} \right) (F_r + \cdots + F_n) = 0 \tag{8}$$

where

$$c_1 = P_m(x_0, y_0) - \frac{x_0}{n} k_{m-1}(x_0, y_0) \qquad c_2 = Q_m(x_0, y_0) - \frac{y_0}{n} k_{m-1}(x_0, y_0) \tag{9}$$

are constants and $N_1(u,v)$, $N_2(u,v)$ are polynomials such that $N_1(0,0) = N_2(0,0) = 0$.

Two cases should be considered:

(1) $c_1 = c_2 = 0$. Then from relations (9) it follows that the equality (5) is fulfilled. Hence $(x_0, y_0, 0) \in D$.

(2) $(c_1, c_2) \neq (0, 0)$. Then one can show from (8) that $c_1 \frac{\partial F_r}{\partial u} + c_2 \frac{\partial F_r}{\partial v} = 0$. Using (7) we see that vectors (c_1, c_2) and (x_0, y_0) are collinear, i.e.

$$\det \begin{pmatrix} c_1 & x_0 \\ c_2 & y_0 \end{pmatrix} = 0$$

which gives again the equality (5).

□

Corollary 1. *Let $D = D_1, \ldots, D_{m+1}$ be a Darboux divisor of the system (1) and $l_i = a_i x + b_i y$, $a_i, b_i \in \mathbb{C}$, $i = 1, \ldots, m+1$, be a set of linear forms such that $l_i(D_i) = 0$, $i = 1, \ldots, m+1$. Then there exist non-negative integers $n_1, \ldots, n_{m+1}$, $\sum n_i = n$ such that*

$$f_n(x, y) = \prod_{i=1}^{m+1} l_i^{n_i}(x, y). \tag{10}$$

Notice that the same expression for f_n was introduced first by Jablonskii [9] in the case $m = 2$, see also [10].

2. The smooth case

Let $C \subset \mathbb{CP}^2$ be an algebraic smooth curve of $\deg(C) = n$ satisfying the equation $f(x, y) = 0$ where $f(x, y)$ is an irreducible algebraic partial integral of the system (1). Without loss of generality we suppose that

$$f(x, y) = y^n + a_1(x) y^{n-1} + \cdots + a_n(x) \qquad a_i(x) \in \mathbb{C}[x] \qquad i = 1, \ldots, n.$$

Consider the holomorphic mapping $\phi : C \to \mathbb{CP}^1$ defined by $\phi(x, y) = x$.

Let $\nu = \nu_\phi(P)$ be a multiplicity of ϕ at the point $P \in C$. Consider the raminification divisor $R = \sum_{p \in C} (\nu_\phi(P) - 1) P \subset \mathrm{Div}(C)$.

We break R into two divisors $R = R_1 + R_2$, where

$$R_1 = \sum_{P \in C \cap L_\infty} (\nu_\phi(P) - 1) P$$

contains branching points of ϕ at infinity and

$$R_2 = \sum_{P \in C / L_\infty} (\nu_\phi(P) - 1) P$$

contains all finite branching points.

Lemma 1. *Let $C = \{f(x, y) = 0\} \subset \mathbb{CP}^2$ be a nonsingular algebraic curve of* $\deg(C) = n$ *where $f(x, y)$ is a partial first integral of the system (1). Then*

$$\deg(R_1) \leqslant n - 1.$$

This statement is proved by noting that $f = f_n + \cdots + f_0$, $\deg f_k = k$ and $f_n = \prod_{i=1}^{m} L_i^{n_i}(x, y)$ where $\sum_{i=1}^{m} n_i = n$, $m \leqslant n$, $L_i(x, y)$ are linear homogeneous polynomials.

Lemma 2.

$$\deg(R_2) = n^2 - n + 1 - \deg(R_1). \tag{11}$$

Proof. Denote $g = \text{genus}(C)$, $n = \deg(C)$, then by the well known formula for a nonsingular curve C we have $g = \frac{(n-1)(n-2)}{2}$.

By the Riemann–Hurwitz formula we obtain $g = \frac{\deg(R)}{2} - n + 1$. Comparing these two expressions for g we find (11). □

Now let us study the divisor R_2.

If $K = (x_0, y_0) \in R_2$ then $\frac{\partial f}{\partial y}(K) = 0$ by the definition of a branching point. With the help of (2) we obtain

$$P(K)\frac{\partial f}{\partial x}(K) + Q(K)\frac{\partial f}{\partial y}(K) = 0. \tag{12}$$

Lemma 3. *If the curve C is nonsingular,* $\deg(C) = n$, *then*

$$\deg(R_2) \leqslant mn$$

where $m > 1$ is the degree of the system (1).

Proof. Since K is a smooth point the relation (12) holds

$$\left(f, \frac{\partial f}{\partial y}\right)_K \leqslant (f, P)_K \qquad K \in R_2$$

where $(g, l)_X$ denotes the intersection number of the curves $g(x, y) = 0$ and $l(x, y) = 0$ at the point $X \in g \cap l$. One can easily verify that $\deg(R_2) = \sum_{P \in R_2} (f, \frac{\partial f}{\partial y})_P$. Thus, by Bézout's theorem $\deg(R_2) \leqslant mn$. □

Theorem 2. *Let us assume that the system (1) admits an smooth algebraic invariant curve $C \subset \mathbb{CP}^2$ defined by the equation $f(x, y) = 0$, $\deg(f) = n$. Then*

$$n \leqslant m + 1$$

where $m > 1$ is the degree of the system (1).

The statement of the theorem follows immediately from the above three lemmas. The theorem 2 was obtained for the first time in [2] using a different method. It was shown by Moulin-Ollagnier that the same result can be obtained in the theory of the Koszul complexes of polynomial vector fields.

3. The Weierstrass polynomials

Let $X = (x_0, y_0) \in \mathbb{C}^2$ be a finite singular point of the curve $C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$, i.e. the point X at which $\frac{\partial f}{\partial x}(X) = \frac{\partial f}{\partial y}(X) = 0$. Without loss of generality we suppose $X = (0, 0)$.

In order to clarify the local structure of C near X, we shall need the help of the *Weierstrass polynomials* [7n7].

Let $\mathbb{C}\{x\}$ ($\mathbb{C}\{x, y\}$) represent the ring of holomorphic functions defined in some neighbourhood of $0 \in \mathbb{C}$ ($(0, 0) \in \mathbb{C}^2$).

Definition 3. *$w \in \mathbb{C}\{x, y\}$ is said to be a Weierstrass polynomial with respect to y, if*

$$w = y^d + c_1(x)y_{d-1} + \cdots + c_d(x) \qquad c_j(x) \in \mathbb{C}\{x\} \qquad c_j(0) = 0 \qquad j = 1, \ldots, d.$$

Let us assume that C is irreducible and its affine equation is

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) = 0.$$

Theorem 3. *The polynomial $f(x, y)$ can be expressed as*

$$f = uf_1 f_2 \cdots f_p$$

where $f_i(x, y) = y^{d_i} + c_{i1}(x)y^{d_i-1} + \cdots + c_{id_i}(x)$, $i = 1, \ldots, p$, are irreducible Weierstrass polynomials and $u(x, y)$ is a unit of $\mathbb{C}\{x, y\}$, i.e. $u(0, 0) \neq 0$.

There exist the open discs $\Delta_i = \{\tau \in \mathbb{C} : |\tau| < \rho_i\}$, $i = 1, \ldots, p$, such that each equation $f_i(x, y) = 0$, $i = 1, \ldots, p$ defines holomorphic mapping $q_i : \Delta_i \to C$ as follows:

$$\tau \to (\tau^{d_i}, g_i(\tau)) \qquad \text{where} \quad g_i(\tau) = \sum_{k=1}^{\infty} c_{ik}\tau^k \in \mathbb{C}\{\tau\} \qquad i = 1, \ldots, p. \tag{13}$$

Thus, from a topological point of view, the algebraic curve C can be obtained near the singular point $X = (0, 0)$ from several open discs by identifying them together at their centres. This is the concept of normalization [7].

Theorem 4. *Let $C = \{(x, y) \in \mathbb{CP}^2 : f(x, y) = 0\}$ be an algebraic invariant curve of the system (1) and $X = (x_0, y_0)$ be a singular point of C. Then X is an equilibrium point of the system (1).*

Proof. Let us assume that $X = (x_0, y_0)$ is not an equilibrium point of the system (1). Then it has the unique solution passing through this point

$$x = x_0 + P(x_0, y_0)t + \sum_{i=2}^{\infty} a_i t^i \qquad y = y_0 + Q(x_0, y_0)t + \sum_{i=2}^{\infty} b_i t^i \qquad a_i, b_i \in \mathbb{C} \tag{14}$$

where $t \in \Delta = \{t \in \mathbb{C} : |t| < \rho\}$ for any small $\rho \in \mathbb{R}$.

On the other hand X is the singular point of C and according to theorem 3 the system (1) has no less than $p > 0$ different solutions passing through X and locally expressed by (13). Thus, we obtain $p = 1$ and the solution (14) is the parametrization of the curve C near the singular point X. By our assumption X is not an equilibrium point of (1), i.e. $P(x_0, y_0) \neq 0$ or $Q(x_0, y_0) \neq 0$. Hence, looking at (14), X is the smooth point of C. We obtain the contradiction. □

Corollary 2. *The number of finite singular points of an arbitrary algebraic invariant curve of the system (1) is not greater than m^2. Furthermore, if $\frac{P_m(x,y)}{Q_m(x,y)} \not\equiv \frac{x}{y}$, then*

$$|\mathrm{Sing}(C)| \leqslant m^2 + m + 1.$$

Indeed, if $\frac{P_m(x,y)}{Q_m(x,y)} \not\equiv \frac{x}{y}$ then the polynomial (3) is not equal to zero identically and according to corollary 1 the curve C cannot have more than $m + 1$ singular points at infinity.

4. The genus of C

Let C be an algebraic invariant curve of the system (1) defined by the equation $f(x, y) = 0$. Denote by $\mathrm{Sing}(C)$ the set of its singular points. There exists the compact Riemann surface $\tilde{C}$ with a surjective continuous map $\pi : \tilde{C} \to C$ such that $\pi : \tilde{C}/\pi^{-1}(\mathrm{Sing}(C)) \to C/\mathrm{Sing}(C)$ is a holomorphic bijection. The aim of this section is to calculate the genus of $\tilde{C}$, which is also called the genus of the curve C. Consider the following meromorphic differential on C:

$$\omega = \frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q}. \tag{15}$$

Let ω be its divisor, then according to the Poincaré–Hopf formula

$$2g - 2 = \deg(\omega). \tag{16}$$

On the other hand, by Noether's formula [8, 11]

$$g = \frac{(n-1)(n-2)}{2} - \sum_{X \in \mathrm{Sing}(C)} \delta(X) \tag{17}$$

where the numbers $\delta(X)$ are given by

$$\delta(X) = \left(f, \frac{\partial f}{\partial y} \right)_X + |\pi^{-1}(X)| - \nu_\phi(X).$$

Here $(\ ,\ )_X$ is the intersection number and $\nu_\phi(X)$ is the multiplicity of the map $\phi : (x, y) \to x$ at the point $(x, y) \in \mathrm{Sing}(C)$.

It is easy to see that ω has no zeros in the affine part of C. Let now $X = (x_0, y_0) \in \mathbb{C}^2$ be the singular point of the curve C. Without loss of generality we put $X = (0, 0)$. According to theorem 3 we can factor $f(x, y)$ into the product of irreducible factors

$$f = u f_1 \cdots f_r$$

where $u(0, 0) \neq 0$ and f_i, $i = 1, \ldots, r$, are Weierstrass polynomials. Notice that $|\pi^{-1}(X)| = r$. Then locally C can be represented as follows:

$$C = C_1 + \cdots + C_r$$

where $C_i = \{(x, y) \in \mathbb{C}^2 : |x| < \rho, |y| < \epsilon, f(x, y) = 0\}$, $\quad i = 1, \ldots, r$ are irreducible local analytic curve components of C and ρ and ϵ are sufficiently small real numbers.

The parametrization of C_i, $i = 1, \ldots, r$, near $X = (0, 0)$ is given by

$$x = \tau^{d_i} \qquad y = \sum_{k=1}^{\infty} c_{ik} \tau^k \qquad c_{ik} \in \mathbb{C} \quad d_i = \deg(f_i). \tag{18}$$

Puting (18) into (15) and using theorem 4 one can show that the differential ω has at the point X a pole of multiplicity at least one. So, for the affine part of the curve C we have the following estimate:

$$\deg(\omega)|_{C \cap \mathbb{C}^2} \leqslant - \sum_{X \in \mathrm{Sing}(C) \cap \mathbb{C}^2} |\pi^{-1}(X)|. \tag{19}$$

Now let us consider the points at infinity. Substituting $x = 1/u$, $y = v/u$ into $f(x, y) = 0$ and multiplying both sides of the resulting expression by u^n, we obtain the equation

$$F(u, v) = f_n(1, v) + u f_{n-1}(1, v) + \cdots + f_0 u^n = 0 \qquad f_0 = \mathrm{const} \neq 0$$

which represents the algebraic curve C near the line at infinity $L_\infty = \{u = 0\}$. We can write $f_n(1, v)$ as follows:

$$f_n(1, v) = \prod_{i=1}^{q} (v - v_i)^{n_i} \qquad n_i = 0, 1, \ldots, q \leqslant n \quad \sum n_i = n \tag{20}$$

where the points $(0, v_i) \in C \cap L_\infty$, $i = 1, \ldots, k$.

Now we break (20) into the product of three factors

$$f_n(1, v) = L_1 L_2 L_3.$$

Here $L_1 = \prod_{i=1}^{r}(v - v_{1i})$, $r \leqslant n$ contains all simple factors of (17). Near the points $(0, v_{1i})$, $i = 1, \ldots, r$, the curve C has the parametrization of the form

$$u = \tau(a_{0i} + a_{1i}\tau + \mathrm{O}(\tau)) \qquad v = v_{1i} + \tau^p(b_{0i} + b_{1i}\tau + \mathrm{O}(\tau)) \tag{21}$$

where $\tau \in \mathbb{C}$ is a local parameter, $a, b \in \mathbb{C}$, $a_{0i} \neq 0$ and p is a positive integer.

$L_2 = \prod_{i=1}^{k}(v - v_{2i})^{m_i}$, $k \leqslant n$ contains factors of multiplicity $m_i > 1$ such that the corresponding points $(0, v_{2i})$ satisfy the condition $\frac{\partial F}{\partial u}(0, v_{2i}) \neq 0$. For arbitrary $1 \leqslant i \leqslant k$ we can write the parametrization of C near $(0, v_{2i})$ as follows:

$$u = \tau^{m_i}(c_{0i} + c_{1i}\tau + \mathrm{O}(\tau)) \qquad v = v_{2i} + \tau(e_{0i} + e_{1i}\tau + \mathrm{O}(\tau)) \qquad c_{0i}, e_{0i} \neq 0. \tag{22}$$

Finally, the factor $L_3 = \prod_{i=1}^{s}(v - v_{3i})^{l_i}$, $s \leqslant n$ includes the multipliers of (20) for which $l_i > 1$ and $\frac{\partial F}{\partial u}(0, v_{3i}) = 0$.

These points are singular and according to theorem 4 near the point $(v_{3i}, 0)$ we have $p_i > 1$ local components of C; each of them can be parametrized as

$$u = \tau^{k_{ij}}(g_{0ij} + g_{1ij}\tau + \mathrm{O}(\tau)) \qquad v = v_{3i} + \tau^{d_{ij}} \qquad g_{0ij} \neq 0 \qquad j = 1, \ldots, p_i \tag{23}$$

where d_{ij}, k_{ij} are positive integers and $\sum_{j=1}^{p_i} k_{ij} \leqslant l_i$.

In addition we have

$$r + \sum_{i=1}^{k} m_i + \sum_{i=1}^{s} l_i = n \qquad \text{and} \qquad C \cap L_\infty = V_1 \cup V_2 \cup V_3$$
$$V_i = \{L_i = 0\} \qquad i = 1, 2, 3.$$

From (15) with the use of (21)–(23) one can show that the following estimates hold:

$$\deg(\omega)|_{V_1} \leqslant r(m-2), \deg(\omega)|_{V_2} \leqslant (m-1)\sum_{i=1}^{k} m_i - k$$
$$\deg(\omega)|_{V_3} \leqslant (m-1)\sum_{i=1}^{s} l_i - \sum_{X \in \mathrm{Sing}(C) \cap L_\infty} |\pi^{-1}(X)|.$$

Summing we obtain

$$\deg(\omega)|_{C \cap L_\infty} \leqslant n(m-1) - \sum_{X \in \mathrm{Sing}(C) \cap L_\infty} |\pi^{-1}(X)| - k - r.$$

Since $\deg(\omega) = \deg(\omega)|_{C \cap L_\infty}(\omega) + \deg(\omega)|_{C \cap \mathbb{C}^2}(\omega)$ in view of (16) and (19) we have the following theorem.

Theorem 5. *For an arbitrary algebraic invariant curve of the system (1) the following estimate for the genus g holds:*

$$2g - 2 \leqslant n(m-1) - \sum_{X \in \mathrm{Sing}(C)} |\pi^{-1}(X)|. \tag{24}$$

This result seems to be a consequence of formula 1 of [2].

5. The algebraic invariant curves with nodes

Let C be an algebraic invariant curve of the system (1) with the defining polynomial $f(x, y)$.

Lemma 4. $|\mathrm{Sing}(C)| \leqslant m^2 + \frac{n}{2}$.

This is a simple consequence of corollary 2 and the notation that C has at most $n/2$ singular points at infinity.

Theorem 6. *Let there exist the integer K such that $\forall\ X \in \mathrm{Sing}(C)$ we have $(f, \frac{\partial f}{\partial y})_X \leqslant K$, then the following estimate for the degree of the curve C holds:*

$$n \leqslant \frac{4 + 2m + K + ((4 + 2m + K)^2 + 16Km^2)^{1/2}}{4} \tag{25}$$

where m is the degree of the system (1).

Proof. Using (17) and (24) one can show that

$$n(n-3) - \sum_{X \in \mathrm{Sing}(C)} \left(f, \frac{\partial f}{\partial y}\right)_X \leqslant n(m-1). \tag{26}$$

By our Assumption, $(f, \frac{\partial f}{\partial y})_X \leqslant K$. According to lemma 4 we obtain immediately

$$\sum_{X \in \mathrm{Sing}(C)} \left(f, \frac{\partial f}{\partial y}\right)_X \leqslant K\left(m^2 + \frac{n}{2}\right). \tag{27}$$

Puting (27) into (26) we arrive at theorem 6. □

Corollary 3. *Let us suppose that all singular points of the algebraic invariant curve C are nodes, then*

$$n \leqslant 2(m+1). \tag{28}$$

Indeed, as a node is an ordinary double point then $K = 1$ and we can use the estimate (25) which gives (28). It is interesting to compare this result with theorem 3 of [2].

Acknowledgment

The author is grateful to L Gavrilov for his attention to this paper and many useful comments.

References

[1] Campillo A and Carnicer M M 1997 Proximity inequalities and bounds for the degree of invariant curves by foliations of P^2_C *Trans. Am. Math. Soc.* **349** 2211–28

[2] Cerveau D and Lins Neto A 1991 Holomorphic foliations in $CP(2)$ having an invariant algebraic curve *Ann. Inst. Fourier Grenoble* **41** 883–903

[3] Chavarriga J and Llibre J 1998 On the algebraic limit cycles of quadratic systems *Proc. 4th Catolan Days of Applied Mathematics (Tarragona, 1998)* (Tarragona: Univ. Rovira Virgili) pp 17–24

[4] Darboux G 1878 Mémoire sur les équations différentielles algébrique du premier ordre et du premier degré (Mélanges) *Bull. Sci. Math.* 60–96

Darboux G 1878 Mémoire sur les équations différentielles algébrique du premier ordre et du premier degré (Mélanges) *Bull. Sci. Math.* 123–44

Darboux G 1878 Mémoire sur les équations différentielles algébrique du premier ordre et du premier degré (Mélanges) *Bull. Sci. Math.* 151–200

[5] Galeotti M 1996 *Monodromic Unbounded Polycycles (Annali di Matematica Pura ed Applicata (IV) vol 171)* pp 83–5

[6] Gonzales-Velasco E 1968 Generic properties of polynomial vector fields at infinity *Trans. Am. Math. Soc.* 143
[7] Griffits P A 1989 *Introduction to Algebraic Curves (Transactions of Mathematical Monographs vol 76)* (Providence, RI: American Mathematical Society)
[8] Hilbert D 1902 Mathematical problems *Bull. Am. Math. Soc.* **8** 437–79
[9] Jablonskii A 1970 Algebraic integrals of system of differential equations *Diff. Uravn.* **6** (11) 1752–60 (in Russian) (Engl. Transl. 1970 1326–33)
[10] Kooij R E and Christopher C J 1993 Algebraic invariant curves and the integrability of polynomial systems *Appl. Math. Lett.* **6** 51–3
[11] Moulin-Ollagnier J, Nowicki A and Strelcyn J-M 1995 On the non-existence of constants of derivations: the proof of theorem of Jouanolou and its development *Bull. Sci. Math.* **119** 195–233
[12] Poincaré H 1928 *Sur les Courbes Définies Par les Équations Différentielles (Oeuvres de Henri Poicaré)* vol 1 (Paris: Gauthiers-Villars)